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A Note on the Beavers and Joseph Condition for Flow over a Forchheimer Porous Layer

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ABSTRACT : *In thiswork, an attemptis made to relate the slip parameter in the Beavers and Joseph condtionat the interface between a Darcy layer and a Navier-Stokes channel, to the slip parameter to beusedwhen the porous layer is a Forchheimer layer.*

Keywords–*Beavers and Joseph condition, Forchheimer porous layer, Slip parameter*

I. INTRODUCTION

In their experimental study of flow through a channel terminated by a Darcy porous layer of semi-infinite depth, Beavers and Joseph [1] observed that the mass flux through the channel was greater than that predicted by Poiseuille flow when a no-slip condition was imposed. Beavers and Joseph, [1], provided an explanation in terms of a slip flow hypothesis at the interface, $y = 0$ (see **Fig. 1** below), and proposed the following empirical slip-flow condition that agreed well with their experiments:

$$\frac{du}{dy} = \frac{\alpha}{\sqrt{k}}(u_B - u_D) \text{ at } y = 0 \ldots(1)$$

where u is the tangential velocity component in the channel, u_D is the uniform Darcy velocity in the porous layer, u_B is the velocity at the interface, k is the constant permeability, and α is an empirical, dimensionless slip coefficient that is independent of fluid viscosity and dependent on the porous medium properties, Reynolds number and flow direction at the interface, [2-7].

Condition (1) has received considerable attention in the porous media literature and has been the subject matter of many elegant investigations, verifications, modifications, or otherwise applications involving flow over various types of porous layers and through composite porous layers, (cf. [8-14] and the references therein).

In case of flow through a channel bounded by a Brinkman layer (which is of compatible differential order to the Navier-Stokes equations), Neale and Nader [13] suggested the use of velocity continuity and shear stress continuity at the interface. They showed that their solution, over a thick porous layer, is the same as the solution obtained using Darcy's equation with Beavers and Joseph condition provided that $\alpha = \sqrt{\mu_{eff} / \mu}$, where μ is the base fluid viscosity and μ_{eff} is the effective viscosity. The effective viscosity of fluid in the porous medium is a semi-empirical quantity, much like the Beavers and Joseph coefficient α. We hasten to point out here that modelling the flow over a Brinkman's porous layer with constant permeability results in permeability discontinuity at the interface. This can be remedied by using Brinkman's equation with variable permeability to serve as a transition layer between a constant permeability layer and the free-space channel (as has been introduced and thoroughly analyzed by Nield and Kuznetsov [6]).

Beavers and Joseph condition, or its modified versions, is important in the flow through a channel underlain by a porous layer the flow through which is governed by an equation of lower order than that of the Navier-Stokes equations (such as a Darcy or a Forchheimer layer). This has been discussed, and recommended by Nield [4], and implemented by Lyubimova*et al*. [14] in their analysis of stability, in connection with a Forchheimer layer. This adds to the already established understanding that in the flow through a channel over a Forchheimer porous layer the use of the Beavers and Joseph condition is appropriate and has been justified. However, this raises an important question with regard to the appropriate value(s) of the coefficient α when the Forchheimer layer is used. In the case of flow over a Darcy layer, Nield [4] provided the range of 0.01 to 5 for α, and reported that in the experiments of Beavers and Joseph, the α values used were 0.78, 1.45, and 4.0 for Foametal having average pore sizes of 0.016, 0.034, and 0.045 inches, respectively, and 0.1 for Aloxite with average pore size of 0.013 or 0.027 inches. Now, if the structure of the porous medium is changed to one where the Forchheimer equation is valid, would the above values be used for the slip parameter?

In order to provide partial answers to this last question, our intention in this work is to use the Beavers and Joseph configuration and assume that the Beavers and Joseph condition is valid in the flow over a Forchheimer layer to find answers to the following equivalent questions:

a) What is the relationship between the slip velocity in the flow over a Darcy layer and the slip velocity in the flow over a Forchheimer porous layer?
b) Since the slip parameter α is empirical and its determination may not be easy, how can existing values of α be modified to find values of a slip parameter to be used in the flow over a Forchheimer layer?

We attempt to answer the above questions in the following scenarios:

(i) Assuming the slip velocities are the same, we find the relationship between the slip parameters when using a Darcy layer and a Forchheimer layer.
(ii) Assuming the slip parameters are the same, what is the relationship between the velocities at the interface when using the two types of posous layers?

II. PROBLEM FORMULATION AND SOLUTION

Consider the Navier-Stokes flow through a channel bounded above by a solid wall, $y = h$, and terminated below by a naturally-occurring semi-infinite porous layer of the type where Forchheimer's equation is valid. The channel and layer intersect at an assumingly sharp porous interface, $y = 0$, as shown in **Fig. 1**.

Flow in the channel is governed by the Navier-Stokes equations, which reduce to the following form:

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \ldots(2)$$

where $u = u(y)$ is the velocity in the channel, p is the pressure, and μ is the viscosity coefficient.

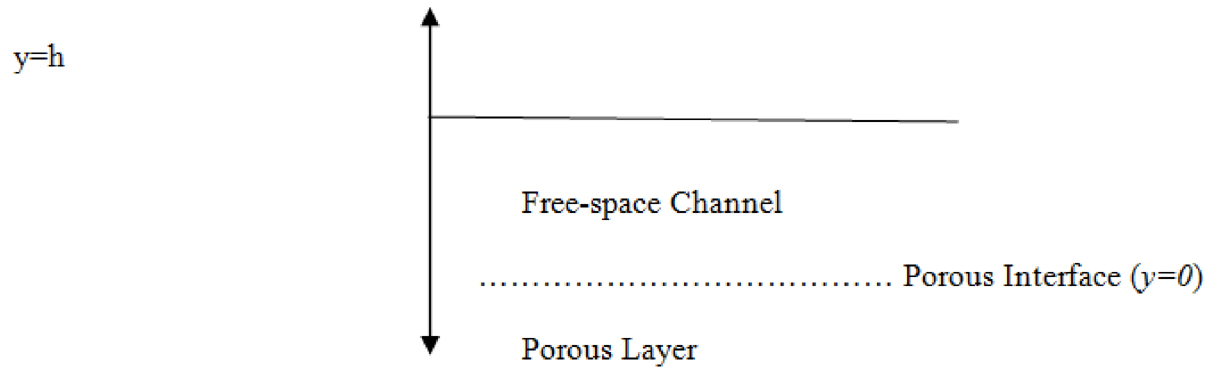

Fig. 1: Diagramatic Sketch of a Channel Bounded by a Porous Layer

Flow through the porous layer is governed by the Forchheimer's equation, of the form:

$$\frac{\rho C_f}{\sqrt{k}} v|v| + \frac{\mu}{k} v + \frac{dp}{dx} = 0 \quad \ldots(3)$$

where v is the velocity in the Forchheimer layer, k is the constant permeability, ρ is the fluid density, and C_f is the Forchheimer drag coefficient. Flow through the given configuration is assumed to be driven by the same constant pressure gradient, $\frac{dp}{dx} < 0$. If $C_f = 0$, equation (3) reduces to Darcy's equation. Furthermore, when all the parameters are constant, equation (3) is an algebraic equation in the velocity, written in the following form, where $\frac{dp}{dx}$ is denoted by p_x:

$$v^2 + \frac{\mu}{\rho C_f \sqrt{k}} v + \frac{\sqrt{k}\, p_x}{\rho C_f} = 0. \quad \ldots(4)$$

Solution to (4) renders the following constant Forchheimer velocity profile across the layer, where the positive root is chosen to obtain a positive velocity, since the quantity under the root is positive for $p_x < 0$:

$$v = -\frac{\mu}{2\rho C_f \sqrt{k}} + \sqrt{\left(\frac{\mu}{2\rho C_f \sqrt{k}}\right)^2 - \frac{\sqrt{k}\, p_x}{\rho C_f}} \,. \quad \ldots(5)$$

Not unlike Darcy's equation, the Forchheimer equation is incompatible with Navier-Stokes equations due to its low differential order. Therefore, solution to equation (2) is sought subject to the no-slip condition, $u = 0$ at $y = h$, and the Beavers-Joseph condition (1) at $y = 0$. Condition (1) is written in the following form in which β replaces α, while u_B has been replaced by u_i and u_D by v in order to distinguish between solutions involving Darcy's equation and those involving Forchheimer's equation:

$$\frac{du}{dy} = \frac{\beta}{\sqrt{k}} (u_i - v) \text{ at } y = 0 \,. \quad \ldots(6)$$

Now, integrating (2) once and employing condition (6), gives:

$$\frac{du}{dy} = \frac{y}{\mu}\frac{dp}{dx} + \frac{\beta}{\sqrt{k}}(u_i - v). \quad \ldots(7)$$

Using (5) in (7) yields

$$\frac{du}{dy} = \frac{y}{\mu}\frac{dp}{dx} + \frac{\beta u_i}{\sqrt{k}} + \frac{\beta\mu}{2\rho k C_f} - \frac{\beta}{2\rho k C_f}\sqrt{[\mu^2 - 4\rho k\sqrt{k} C_f p_x]}\,. \quad \ldots(8)$$

By comparison, Darcy's case [1] yields:

$$\frac{du}{dy} = \frac{y}{\mu}\frac{dp}{dx} + \frac{\alpha}{\sqrt{k}} u_i + \alpha\frac{\sqrt{k}}{\mu}\frac{dp}{dx}\,. \quad \ldots(9)$$

Now, integrating (8) and using the condition $u(h) = 0$ yields

$$u = \frac{(y^2 - h^2)}{2\mu}\frac{dp}{dx} + \left\{\frac{\beta u_i}{\sqrt{k}} + \frac{\beta\mu}{2\rho k C_f} - \frac{\beta}{2\rho k C_f}\sqrt{[\mu^2 - 4\rho k\sqrt{k} C_f p_x]}\right\}(y - h)\,. \quad \ldots(10)$$

Evaluating (10) at $y = 0$ and solving for velocity at the interface, u_i, gives

$$u_i = -\frac{k}{2\mu}\left[\frac{\sigma^2 + 2\beta\sigma\xi}{1 + \beta\sigma}\right]\frac{dp}{dx} \quad \ldots(11)$$

where

$$\xi = \frac{\mu}{2\rho C_f k\sqrt{k}\, p_x}\left\{\mu - \sqrt{[\mu^2 - 4\rho k\sqrt{k} C_f p_x]}\right\} \quad \ldots(12)$$

and

$$\sigma = \frac{h}{\sqrt{k}}\,. \quad \ldots(13)$$

By comparison, when Darcy's equation governs the flow in the porous layer, velocity at the interface is given by, [1]

$$u_B = \frac{-k}{2\mu}\left\{\frac{\sigma^2 + 2\alpha\sigma}{1 + \alpha\sigma}\right\}\frac{dp}{dx}\,. \quad \ldots(14)$$

Equation (12) implies that $\xi \neq 0$ and is a positive quantity for non-zero flow parameters. When $\xi = 1$, equations (11) and (14) are the same. Alternatively, as the Forchheimer velocity approaches the Darcy velocity (for small Reynolds number), the value of ξ approaches unity.

In using either the Darcy or the Forchheimer porous layer if we assume that the permeabilites are equal, so are the pressure gradients and the fluid viscosities, then (11) and (14) yield the following relationship between the velocities at the interface:

$$u_i = \frac{(\sigma+2\beta\xi)(1+\alpha\sigma)}{(\sigma+2\alpha)(1+\beta\sigma)}u_B = \frac{(h+2\beta\xi\sqrt{k})}{(h+2\alpha\sqrt{k})}\frac{(\sqrt{k}+\alpha h)}{(\sqrt{k}+\beta h)}u_B . \quad \ldots(15)$$

Furthermore, if $u_i = u_B$ then

$$\beta = \frac{\alpha[\sigma^2-2]}{[\sigma^2+2\alpha\sigma(1-\xi)-2\xi]} = \frac{\alpha(2k-h^2)}{\{2\xi k+2(\xi-1)\alpha h\sqrt{k}-h^2\}} . \quad \ldots(16)$$

If $\alpha = \beta$ then the velocities at the interface in the two cases are related by

$$u_i = \frac{(\sigma+2\alpha\xi)}{(\sigma+2\alpha)}u_B = \frac{(h+2\alpha\xi\sqrt{k})}{(h+2\alpha\sqrt{k})}u_B \ldots(17)$$

Equations (14) and (17) imply that both u_B and u_i are positive velocities. Beavers and Joseph [1] argued that h is *"considerably larger than* $\sqrt{2k}$ *"* otherwise the *"assumption of rectilinear flow in the channel breaks down"*. This implies that, in equation (16), $\alpha(2k-h^2)<0$. In order to have $\beta>0$ we must have $2\xi k+2(\xi-1)\alpha h\sqrt{k}-h^2<0$. This implies that $\xi < \frac{2\alpha h\sqrt{k}+h^2}{2\alpha h\sqrt{k}+2k}$.

III. COMPUTATIONS AND RESULTS

Nield [4] provided detailed analysis of the Beavers and Joseph condition, provided a range of 0.01 to 5 for the slip parameter, and reported the following values used in the experiments of Beavers and Joseph for α : 0.1, 0.78, 1.45, and 4.0.We will utilize these values of α in computing values for β. For the sake of current computations, we use dimensionless forms of the flow and domain quantities with respect to a characteristic velocity u_c and channel depth *h*, using the following definitions in which Re is Reynolds number:

$$Y = \frac{y}{h};(U,V) = \frac{(u,v)}{u_c};K = \frac{k}{h^2};P = \frac{p}{\rho(u_c)^2};\mathrm{Re} = \frac{\rho u_c h}{\mu} . \quad \ldots(18)$$

In dimensionless form, P_X is the dimensionless pressure gradient, $\sigma = \frac{1}{\sqrt{K}}$ and

$$\xi = \frac{1-\sqrt{[1-4K\sqrt{K}C_f\,\mathrm{Re}^2 P_X]}}{2C_f K\sqrt{K}\,\mathrm{Re}^2 P_X} = \frac{2}{1+\sqrt{[1-4K\sqrt{K}C_f\,\mathrm{Re}^2 P_X]}} . \quad \ldots(19)$$

It is clear from (19) that for small Re or small dimensionless permeability, ξ approaches unity. The behavior of ξ for different values of dimensionless permeability and Reynolds number is illustrated in **Fig. 2**, below, and shows how ξ approaches unity. Changes in ξ as a function of dimensionless permeability for different dimensionless pressure gradients are illustrated in **Fig. 3**.

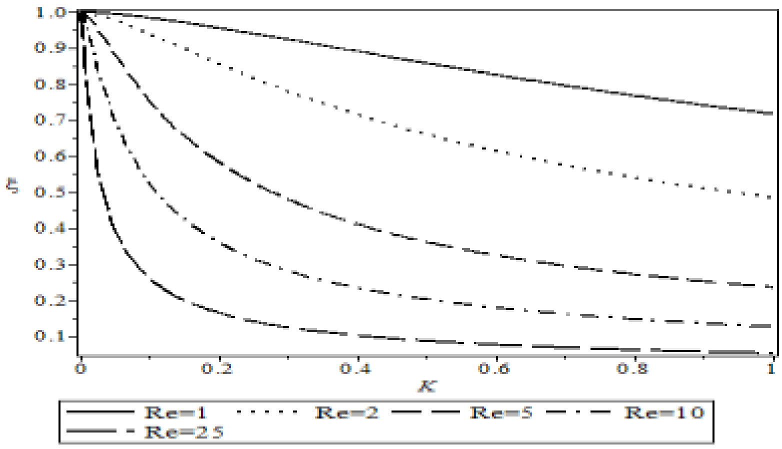

Fig. 2: Graphs of ξ vs.dimensionless permeability for different values of Re.

$P_X = -1$ and $C_f = 0.55$

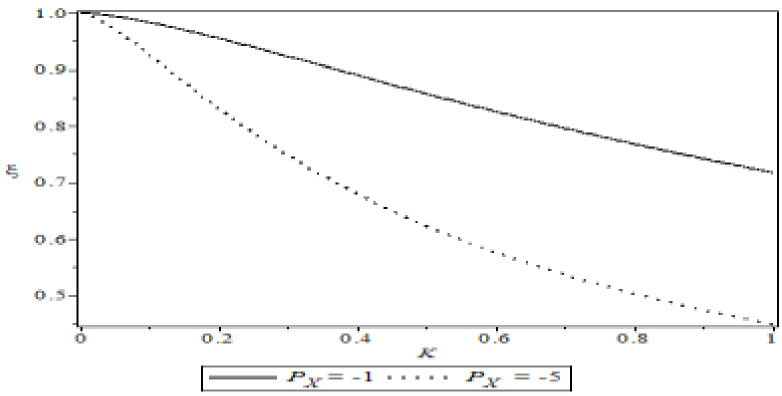

Fig. 3 Graphs of ξ as a function of peameabilityfor different values of P_X.

Re=1, $C_f = 0.55$.

Dimensionless velocities at the interface are obtained from (11) and (14), respectively, as:

$$U_i = -\frac{K \operatorname{Re} P_X}{2}\left[\frac{\sigma^2 + 2\beta\sigma\xi}{1+\beta\sigma}\right] = -\frac{\operatorname{Re} P_X}{2}\left[\frac{\sqrt{K} + 2\beta K\xi}{\sqrt{K}+\beta}\right] \ldots(20)$$

$$U_B = \frac{-K \operatorname{Re} P_X}{2}\left\{\frac{\sigma^2 + 2\alpha\sigma}{1+\alpha\sigma}\right\} = \frac{-\operatorname{Re} P_X}{2}\left\{\frac{1+2\alpha K}{\sqrt{K}+\alpha}\right\}. \quad \ldots(21)$$

Using (20) and (21), the following relationship between these velocities is obtained:

$$U_i = \frac{(\sigma+2\beta\xi)}{(\sigma+2\alpha)}\frac{(1+\alpha\sigma)}{(1+\beta\sigma)}U_B = \frac{(1+2\beta\sqrt{K}\xi)}{(1+2\alpha\sqrt{K})}\frac{(\sqrt{K}+\alpha)}{(\sqrt{K}+\beta)}U_B. \quad \ldots(22)$$

If $\alpha = \beta$ then (22) yields:

$$U_i = \frac{(\sigma+2\alpha\xi)}{(\sigma+2\alpha)}U_B = \frac{(1+2\alpha\sqrt{K}\xi)}{(1+2\alpha\sqrt{K})}U_B. \quad \ldots(23)$$

The ratio of dimensionless velocities at the interface, U_i / U_B, obtained from equation (23) is plotted against dimensionless permeability, *K*, for the range of values of α in **Fig. 4**, which shows an increase in the ratio with increasing α and with increasing dimensionless permeability. Over the range $0 < K < 1$ the ratio U_i / U_B is greater than 0.75 and less than unity, thus indicating that the Forchheimer velocity at the interface, U_i, is always less than the Darcy velocity at the interface, U_B.

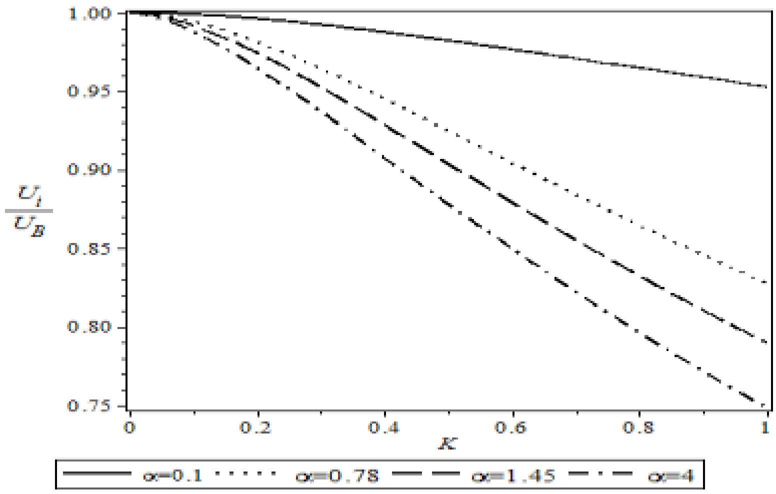

Fig. 4. The ratio U_i / U_B for Re=1, P_X =-1, and different values of α

If $U_i = U_B$ the following relationship between α and β is obtained:

$$\beta = \frac{\alpha(2-\sigma^2)}{(2\xi - \sigma^2) + 2\alpha\sigma(\xi - 1)} = \frac{\alpha(2K-1)}{(2K\xi - 1) + 2\alpha\sqrt{K}(\xi - 1)}. \quad \ldots(24)$$

Since α and β are positive therefore $K \neq 0.5$. For $K < 0.5$, the assumption of rectilinear flow in the channel holds, and the choices of ξ and α must satisfy $(2K\xi - 1) + 2\alpha\sqrt{K}(\xi - 1) < 0$.

Values of ξ, computed using equation (19), and the subsequent computing of values of β using equation (24) are tabulated in **Tables 1(a)** and **1(b)**, which demonstrate that the values of β approach the values of α as the dimensionless permeability gets smaller, for the tested dimensionless pressure gradients and Reynolds number. For a given small dimensionless permeability, β gets closer to α with decreasing Re.

Table 2(a). Values of β corresponding to values of α, for different values K; Re = 1, $C_f = 0.55$, $P_X = -1$

K	ξ	α	β
0.499	0.8574588945	0.1	0.00121658785
0.499	0.8574588945	0.78	0.00517698159
0.499	0.8574588945	1.45	0.00664741591
0.499	0.8574588945	4.0	0.00842297179
0.01	0.999450909	0.1	0.09999775887
0.01	0.999450909	0.78	0.77992309
0.01	0.999450909	1.45	1.44974819
0.01	0.999450909	4.0	3.99816307
0.000001	0.999999999	0.1	0.09999998

0.000001	0.999999999	0.78	0.77999985
0.000001	0.999999999	1.45	1.449999736
0.000001	0.999999999	4.0	3.999999273

Table 2(b). Values of β corresponding to values of α, for different values *K; Re* = 5, $C_f = 0.55$, $P_X = -1$

K	ξ	α	β
0.499	0.3626334588	0.1	0.000274672
0.499	0.3626334588	0.78	0.00116378
0.499	0.3626334588	1.45	0.001491942
0.499	0.3626334588	4.0	0.001886803
0.01	0.9866156	0.1	0.0999453996
0.01	0.9866156	0.78	0.7781295874
0.01	0.9866156	1.45	1.443886813
0.01	0.9866156	4.0	3.955699364
0.000001	0.99999998	0.1	0.099999996
0.000001	0.99999998	0.78	0.7799999716
0.000001	0.99999998	1.45	1.449999947
0.000001	0.99999998	4.0	3.999999854

IV. CONCLUSION

In this work we provided analysis of the Beavers and Joseph condition when applied to the interface between a Forchheimer porous layer and a Navier-Stokes channel. We derived expressions relating the slip parameters in a Darcy layer and a Forchheimer layer under the assumption of equal interfacial velocities, and derived an expression for the ratio of the velocities at the interface when the slip coefficient in the Forchheimer layer is the same as that in the Darcy layer.

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